Problem 1.22

There is a marvelous problem, the Snowplow Problem of R. P. Agnew, which reads, "One day it started snowing at a heavy and steady rate. A snowplow started out at noon, going 2 miles the first hour and 1 mile the second hour. What time did it start snowing?" (*Answer:* 11:23 A.M.) We also recommend a more sophisticated variation by M. S. Klamkin called the Great Snowplow Chase: "One day it started snowing at a heavy and steady rate. Three identical snowplows started out at noon, 1 P.M., and 2 P.M. from the same place and all collided at the same time. What time did it start snowing?" (*Answer:* 11:30 A.M.)

Clue: The speed of the plow is inversely proportional to the height of the snow.

Solution

The Snowplow Problem of R. P. Agnew

From the first sentence we can deduce the height of the snow at time t. Let it be denoted as $h(t)$. Because it's snowing at a steady rate, the rate that the height increases must be constant. Call it H.

$$\frac{dh}{dt} = H$$

Integrate both sides of this equation with respect to t to solve for $h(t)$.

$$h(t) = Ht + C,$$

where C is an arbitrary constant. When $t = 0$, it's noon, and that's the time when the snowplow starts moving. Let T represent the amount of time before noon that it started to snow. Then at noon the height of the snow must be HT, that is, $h(0) = HT$. We can use this initial condition to solve for C.

$$h(0) = C = HT$$

Hence,

$$h(t) = H(t + T).$$

Our goal for this problem is to solve for T. The fact that the snowplow travels 2 miles the first hour and only 1 mile the second hour means that its speed is changing. The speed v is defined as the rate that the position x changes with respect to time.

$$v = \frac{dx}{dt}$$

Because we're given information about the position of the snowplow, solve this equation for x by integrating both sides.

$$x = \int v(t)\,dt$$

With this we can write two equations from the given information.

$$2 = \int_0^1 v(t)\,dt \tag{1}$$

$$1 = \int_1^2 v(t)\,dt \tag{2}$$

According to the clue, the speed of the snowplow is inversely proportional to the height of the snow. We can write a proportionality from this.

$$v(t) \propto \frac{1}{h(t)}$$

To change this to an equation we can use, we introduce a constant of proportionality k. Also, substitute the expression we found for $h(t)$.

$$v(t) = \frac{k}{H(t+T)}$$

With this formula for $v(t)$, equations (1) and (2) become the following.

$$\begin{aligned} 2 &= \int_0^1 \frac{k}{H(t+T)}\,dt \\ 1 &= \int_1^2 \frac{k}{H(t+T)}\,dt \end{aligned}$$

Despite the fact that there are three unknowns (k, H, and T) and only two equations, we can solve for what we want since we can treat k/H as one variable and eliminate it. Before we do that, though, evaluate the integrals.

$$\begin{aligned} 2 &= \frac{k}{H}\ \ln(t+T)\bigg|_0^1 = \frac{k}{H}\left[\ln(1+T) - \ln T\right] = \frac{k}{H}\ln\frac{1+T}{T} \\ 1 &= \frac{k}{H}\ \ln(t+T)\bigg|_1^2 = \frac{k}{H}\left[\ln(2+T) - \ln(1+T)\right] = \frac{k}{H}\ln\frac{2+T}{1+T} \end{aligned}$$

To eliminate k/H, divide the first equation by the second one.

$$2 = \frac{\ln\dfrac{1+T}{T}}{\ln\dfrac{2+T}{1+T}}$$

From this point on it's just a matter of algebra to solve for T.

$$\begin{aligned} 2\ln\frac{T+2}{T+1} &= \ln\frac{T+1}{T} \\ \ln\left(\frac{T+2}{T+1}\right)^2 &= \ln\frac{T+1}{T} \\ \left(\frac{T+2}{T+1}\right)^2 &= \frac{T+1}{T} \\ \frac{T^2+4T+4}{T^2+2T+1} &= \frac{T+1}{T} \\ \cancel{T^3} + 4T^2 + 4T &= \cancel{T^3} + 2T^2 + T + T^2 + 2T + 1 \\ 4T^2 + 4T &= 3T^2 + 3T + 1 \\ T^2 + T - 1 &= 0 \\ T &= \frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

Since T has to be a positive number, we choose the plus sign.

$$T = \frac{-1 + \sqrt{5}}{2} \approx 0.618$$

t is in units of hours, so T is in hours as well. To convert this to minutes, multiply the result by 60. $0.618 \times 60 \approx 37.1$ minutes. That is, it started to snow about 37 minutes prior to noon. Therefore, it started snowing at about 11:23 A.M.

The Great Snowplow Chase of M. S. Klamkin

For this problem there are three snowplows, each with its own speed that depends on the height of the snow in front of it. The analysis we did in the previous problem holds for the first snowplow set into motion at noon. The two snowplows behind it do not influence its motion. If we denote its position as x_1, the time to get to that position as t_1, its speed as v_1, and the height of snow in front of it as h_1, then we have

$$v_1 = \frac{k}{h_1} = \frac{k}{H(t_1 + T)},$$

the same expression as before but with subscripts. Like before, our aim is to solve for T, the time before noon that it started to snow. We will use similar variables for the second and third snowplows—x_2, t_2, v_2, and h_2 for the second snowplow and x_3, t_3, v_3, and h_3 for the third snowplow. The corresponding equations for the speeds of the second and third snowplow are

$$\begin{aligned} v_2 &= \frac{k}{h_2} \\ v_3 &= \frac{k}{h_3}. \end{aligned}$$

When the first snowplow goes past some point x_1 at time t_1, the height of the snow there is set to 0. When the second snowplow comes to that same point at time t_2, the height of the snow there is the rate that snow builds up multiplied by the time elapsed since the first snowplow was there. That is, $h_2 = H(t_2 - t_1)$. The story is the same for h_3. For a certain point x_2 that the second snowplow passes at t_2, the height of snow that the third snowplow will come to at t_3 will be H multiplied by the time elapsed since the second snowplow was there. That is, $h_3 = H(t_3 - t_2)$. Thus, we have the following system of equations for the speeds.

$$\begin{aligned} v_1 &= \frac{k}{H(t_1 + T)} \\ v_2 &= \frac{k}{H(t_2 - t_1)} \\ v_3 &= \frac{k}{H(t_3 - t_2)} \end{aligned}$$

Speed is the rate that the position changes with respect to time.

$$v = \frac{dx}{dt}$$

Since the right sides of the system are in terms of t_1, t_2, and t_3, we choose these to be the dependent variables and x to be the independent variable.

$$\begin{aligned} v_1 &= \frac{dx}{dt_1} = \frac{k}{H(t_1 + T)} \\ v_2 &= \frac{dx}{dt_2} = \frac{k}{H(t_2 - t_1)} \\ v_3 &= \frac{dx}{dt_3} = \frac{k}{H(t_3 - t_2)} \end{aligned}$$

Since we're solving for t_1, t_2, and t_3, invert both sides of each equation and let $A = H/k$ to make the following solution less messy.

$$\begin{aligned} \frac{dt_1}{dx} &= \frac{H}{k}(t_1 + T) = A(t_1 + T) \\ \frac{dt_2}{dx} &= \frac{H}{k}(t_2 - t_1) = A(t_2 - t_1) \\ \frac{dt_3}{dx} &= \frac{H}{k}(t_3 - t_2) = A(t_3 - t_2) \end{aligned}$$

The first, second, and third snowplows are set in motion at the same spot, $x = 0$, at noon, 1 P.M., and 2 P.M., respectively, so the initial conditions are $t_1(x = 0) = 0$, $t_2(x = 0) = 1$, and $t_3(x = 0) = 2$. The first equation is a first-order inhomogeneous ODE for t_1, so it can be solved by multiplying both sides by an integrating factor.

$$\frac{dt_1}{dx} - At_1 = AT$$

The integrating factor is

$$I = e^{\int^x -A\,ds} = e^{-Ax}.$$

Multiplying both sides by this gives us

$$e^{-Ax}\frac{dt_1}{dx} - Ae^{-Ax}t_1 = ATe^{-Ax}.$$

The left side is now exact and can be written as $d/dx(It_1)$ as a result of the product rule.

$$\frac{d}{dx}(e^{-Ax}t_1) = ATe^{-Ax}$$

Integrate both sides with respect to x.

$$e^{-Ax}t_1 = -Te^{-Ax} + C_1$$

Multiply both sides by e^{Ax} to get

$$t_1(x) = -T + C_1e^{Ax}.$$

Plug in the initial condition now to determine the constant C_1.

$$t_1(0) = -T + C_1 = 0 \quad \rightarrow \quad C_1 = T$$

Thus, we have the solution for t_1.

$$t_1(x) = T(e^{Ax} - 1)$$

Now we're in a position to determine t_2.

$$\frac{dt_2}{dx} = A(t_2 - t_1) = At_2 - AT(e^{Ax} - 1)$$

Again, this is a first-order inhomogeneous equation that can be solved by multiplying both sides by an integrating factor.

$$\frac{dt_2}{dx} - At_2 = -ATe^{Ax} + AT$$

The integrating factor is

$$I = e^{\int^x -A\,ds} = e^{-Ax}.$$

Multiplying both sides of the equation gives us

$$e^{-Ax}\frac{dt_2}{dx} - Ae^{-Ax}t_2 = -AT + ATe^{-Ax}.$$

The left side is now exact and can be written as $d/dx(It_2)$ as a result of the product rule.

$$\frac{d}{dx}(e^{-Ax}t_2) = -AT + ATe^{-Ax}$$

Integrate both sides with respect to x.

$$e^{-Ax}t_2 = -ATx - Te^{-Ax} + C_2$$

Multiply both sides by e^{Ax}.

$$t_2(x) = -ATxe^{Ax} - T + C_2e^{Ax}$$

Plug in the initial condition now to determine the constant C_2.

$$t_2(0) = -T + C_2 = 1 \quad \rightarrow \quad C_2 = 1 + T$$

Thus, we have the solution for t_2.

$$t_2(x) = e^{Ax}(1 + T - ATx) - T$$

Now we're in a position to solve for t_3.

$$\frac{dt_3}{dx} = A(t_3 - t_2) = At_3 - Ae^{Ax}(1 + T - ATx) + AT$$

Again, this is a first-order inhomogeneous equation that can be solved by multiplying both sides by an integrating factor.

$$\frac{dt_3}{dx} - At_3 = -Ae^{Ax} - ATe^{Ax} + A^2Txe^{Ax} + AT$$

The integrating factor is

$$I = e^{\int^x -A\,ds} = e^{-Ax}.$$

Multiplying both sides of the equation gives us

$$e^{-Ax}\frac{dt_3}{dx} - Ae^{-Ax}t_3 = -A - AT + A^2Tx + ATe^{-Ax}.$$

The left side is now exact and can be written as $d/dx(It_3)$ as a result of the product rule.

$$\frac{d}{dx}(e^{-Ax}t_3) = -A - AT + A^2Tx + ATe^{-Ax}$$

Integrate both sides with respect to x.

$$e^{-Ax}t_3 = -Ax - ATx + \frac{1}{2}A^2Tx^2 - Te^{-Ax} + C_3$$

Multiply both sides by e^{Ax}.

$$t_3(x) = -A(1+T)xe^{Ax} + \frac{1}{2}A^2Tx^2e^{Ax} - T + C_3e^{Ax}$$

Plug in the initial condition now to determine the constant C_3.

$$t_3(0) = -T + C_3 = 2 \quad \rightarrow \quad C_3 = 2 + T$$

Thus, we have the solution for t_3.

$$t_3(x) = e^{Ax}\left\{2 - Ax + \frac{T}{2}[Ax(Ax-2)+2]\right\} - T$$

Now that we are armed with $t_1(x)$, $t_2(x)$, and $t_3(x)$, we're ready to consider the last part of the question. When all three snowplows collide, they are at the same final position X_F at the same time T_F. So guess what? We have yet another system of equations to solve!

$$\begin{aligned} t_1(X_F) &= T_F \\ t_2(X_F) &= T_F \\ t_3(X_F) &= T_F, \end{aligned}$$

which is

$$\begin{aligned} T(e^{AX_F} - 1) &= T_F \\ e^{AX_F}(1 + T - ATX_F) - T &= T_F \\ e^{AX_F}\left\{2 - AX_F + \frac{T}{2}[AX_F(AX_F-2)+2]\right\} - T &= T_F. \end{aligned}$$

In order to substitute the first equation into the other two, write the second and third equations like so.

$$\begin{aligned} T(e^{AX_F} - 1) &= T_F \\ T(e^{AX_F} - 1) + e^{AX_F}(1 - ATX_F) &= T_F \\ T(e^{AX_F} - 1) + e^{AX_F}\left[2 - AX_F + \frac{ATX_F}{2}(AX_F - 2)\right] &= T_F \end{aligned}$$

Now substitute the first equation into the other two and then cancel T_F from both sides.

$$\begin{aligned} e^{AX_F}(1 - ATX_F) &= 0 \\ e^{AX_F}\left[2 - AX_F + \frac{ATX_F}{2}(AX_F - 2)\right] &= 0 \end{aligned}$$

Divide both sides of each equation by e^{AX_F}.

$$\begin{aligned} 1 - ATX_F = 0 \quad &\rightarrow \quad AX_F = \frac{1}{T} \\ 2 - AX_F + \frac{ATX_F}{2}(AX_F - 2) = 0 & \end{aligned}$$

Finally, substitute $AX_F = 1/T$ into the second equation to get an equation for T.

$$2 - \frac{1}{T} + \frac{1}{2}\left(\frac{1}{T} - 2\right) = 0$$

Multiply both sides by the least common denominator.

$$4T - 2 + 1 - 2T = 0$$

$$2T - 1 = 0$$

$$T = \frac{1}{2}$$

Since t is expressed in hours, T is in hours as well. To convert this to minutes, multiply the result by 60. $1/2 \times 60 = 30$ minutes. That is, it started to snow 30 minutes prior to noon. Therefore, it started snowing at 11:30 A.M.

Just as a final note, we can solve the last system of equations for the time that the snowplows all collide T_F. We get

$$T_F = \frac{1}{2}(e^2 - 1) \approx 3.195.$$

Hence, the snowplows collide 3.195 hours after noon, or at about 3:12 P.M.